Equations Involving the Partial Derivatives of a Function of a Surface.[*]

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Introduction.

In a former paper[†] I have defined the derivative of a function of a surface, and proved that if a function $L(S)$ has a derivative $L'(S;x,y)$, which is continuous and approached uniformly, the first variation of $L(S)$ can be given by the equation

$$\frac{dL(S)}{d\lambda}=\int\int L'(S;x,y)\frac{dz(x,y)}{d\lambda}dxdy. \tag{1}$$

Conversely, it can be proved that if there is a function $L'(S;x,y)$, continuous in all its arguments, which satisfies this equation for every family of surfaces in a given neighborhood, then $L'(S;x,y)$ is the derivative of $L(S)$. In the present paper, functions depending not only on a surface, but also on the values taken by a function at every point of the surface, are considered. Such a function has two partial functional derivatives.

In the first section these derivatives are defined. In Section 2 the adjoint of a functional is discussed, and the conditions that functionals of a certain type be self-adjoint are found. In Section 3 are found the conditions that two given functionals be the partial derivatives of a function of a surface. The fourth section contains the condition of integrability for an equation involving these derivatives, and the equation is found which is satisfied by the function

$$\Phi=\int\int\int(f_x^2+f_y^2+f_z^2)\,dxdydz,$$

where $f(x,y,z)$ is a potential function. In the last section the characteristics of such an equation are briefly discussed. Similar work for functions of lines has been done by Levy.[‡]

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* Read before the American Mathematical Society, Feb. 26, 1916.

† American Journal of Mathematics, Vol. XXXVI (1914), No. 3, p. 289.

‡ "Sur l'integration des equations aux derivees fonctionelle partielles," *Rendiconti del Circolo Matematico di Palermo*, Vol. 37 (1914), p. 113.

§ 1. *The Variation of a Function of a Surface.*

The equations of surfaces discussed in this paper will be given in parametric form; and, consequently, the definition of the derivative of a function of a surface must be modified slightly. The equations of the surface S will be

$$S: \qquad x=x(u,v), \quad y=y(u,v), \quad z=z(u,v),$$

defined over a region Ω in the u, v-plane, and those of the varied surface,

$$\begin{aligned} S_\epsilon: \qquad & x=x(u,v)+Xn(u,v), \quad y=y(u,v)+Yn(u,v), \\ & z=z(u,v)+Zn(u,v), \end{aligned}$$

where X, Y and Z are the direction cosines of the normal to S. The function $n(u,v)$ is assumed to be of class $C^{(r)}$, to have a permanent sign, to vanish everywhere excepting in the region $(u_0-\varepsilon<u<u_0+\varepsilon;\ v_0-\varepsilon<v<v_0+\varepsilon)$, and the absolute values of its partial derivatives including those of order r are assumed to be less than ε. Then the derivative said to be approached with order r, will be defined as

$$L'(S;u,v)=\lim_{\epsilon=0}\frac{L(S_\epsilon)-L(S)}{\sigma},$$

where

$$\sigma=\int_{u_0-\epsilon}^{u_0+\epsilon}\int_{v_0-\epsilon}^{v_0+\epsilon} n(u,v)\,H(u,v)\,dudv, \tag{2}$$

$Hdudv$ being the element of area. That is, $H=\sqrt{EG-F^2}$, where $E=\Sigma x_u^2$, $F=\Sigma x_u x_v$ and $G=\Sigma x_v^2$, the summation sign signifying that the expression is symmetrical in x, y and z. It will always be assumed that $H\neq 0$. If the function $\omega(x,y,\alpha)$ in my paper already referred to,* is replaced by the new function $n(u,v,\lambda)$, equation (9) in that paper can be replaced by the equation,

$$\left.\frac{dL(S_\lambda)}{d\lambda}\right|_{\lambda=0}=\iint_\Omega L'(S;u,v)\,n_\lambda(u,v,0)\,Hdudv. \tag{3}$$

In order to make the variation normal to all of the surfaces S_λ, instead of to S_0 alone, these surfaces can be considered as defined by the partial differential equations

$$S_\lambda: \qquad x_\lambda=Xn_\lambda, \quad y_\lambda=Yn_\lambda, \quad z_\lambda=Zn_\lambda, \tag{4}$$

with the initial conditions $x(u,v,0)=x(u,v)$, $y(u,v,0)=y(u,v)$, $z(u,v,0)=z(u,v)$. The existence theorems for partial differential equations prove that $x(u,v,\lambda)$, $y(u,v,\lambda)$ and $z(u,v,\lambda)$ are determined uniquely, at least if

* Fischer, *loc. cit.*, p. 291.

$n(u, v, \lambda)$, $x(u, v)$, $y(u, v)$ and $z(u, v)$ are analytic, and cases where they are not so determined will not be considered here. This change in the equations of S_λ does not affect equation (3).

A function $\Phi(n, f)$ will now be considered which depends on the surface designated by the argument n, and on all of the values taken by another function $f(u, v)$ at points of the surface. Such a function may have two partial functional derivatives,

$$\Phi'_n(0, f; u, v) = \lim_{\epsilon=0} \frac{\Phi(n, f) - \Phi(0, f)}{\sigma}, \tag{5}$$

and

$$\Phi'_f(n, f_0; u, v) = \lim_{\epsilon=0} \frac{\Phi(n, f) - \Phi(n, f_0)}{\sigma'}, \tag{6}$$

where σ is defined by equation (2), and

$$\sigma' = \int_{u_0-\epsilon}^{u_0+\epsilon} \int_{v_0-\epsilon}^{v_0+\epsilon} (f(u, v) - f_0(u, v)) H du dv.$$

It is assumed, of course, that $f - f_0$ has the properties already assumed for n. If these partial derivatives are continuous in all arguments and approached uniformly, the equation

$$\frac{d\Phi(n, f)}{d\lambda} = \iint_\Omega [\Phi'_n(n, f; u, v) n_\lambda + \Phi'_f(n, f; u, v) f_\lambda] H du dv, \tag{7}$$

which is analogous to equation (3), must be satisfied, if $n(u, v, \lambda)$ and $f(u, v, \lambda)$ and all of their partial derivatives considered are continuous everywhere, and independent of λ along the boundary of Ω, and if $n_\lambda(u, v, \lambda)$ and $f_\lambda(u, v, \lambda)$ are approached uniformly. It was also assumed to simplify the proof that the function which corresponds to $n(u, v, \lambda') - n(u, v, \lambda)$ should have a permanent sign,* but this is not essential.

§ 2. *The Adjoint of a Functional.*

The adjoint of a functional of a line has been defined by Levy,† and it was found useful in deriving the condition of integrability for equations involving functional derivatives. The adjoint of a functional of a surface will be defined similarly.

If there are two functionals $L(f)$ and $\overline{L}(f)$ such that the equation

$$\iint_\Omega g(u, v) L(f) H du dv = \iint_\Omega f(u, v) \overline{L}(g) H du dv \tag{8}$$

* Fischer, *loc, cit.*, p. 291.

† Levy, *loc. cit.*, p. 115.

is satisfied for every pair of functions f and g of class $C^{(r)}$ which vanish together with their partial derivatives along the boundary of Ω, then $\overline{L}$ is said to be the adjoint of L, and vice versa. The proof that a functional can not have two distinct adjoints is essentially the same as for functionals of lines,* and will not be repeated. If a functional has an adjoint it must be linear, since if $\overline{L}$ is the adjoint of L the equations

$$\begin{aligned}\iint_\Omega gL(af_1+bf_2)Hdudv &= a\iint_\Omega f_1\overline{L}(g)Hdudv+b\iint_\Omega f_2\overline{L}(g)Hdudv\\ &= \iint g[aL(f_1)+bL(f_2)]Hdudv,\end{aligned}$$

must be satisfied for every function $g(u, v)$. It follows that

$$L(af_1+bf_2)=aL(f_1)+bL(f_2),$$

which is the condition that L be linear. Every linear functional can be expressed in the form

$$\lim_{\epsilon=0}\iint_\Omega F(u, v, u_1, v_1, \epsilon)f(u_1, v_1)H(u_1, v_1)du_1dv_1.\dagger$$

Its adjoint will be

$$\lim_{\epsilon=0}\iint_\Omega F(u_1, v_1, u, v, \epsilon)f(u_1, v_1)H(u_1, v_1)du_1dv_1,$$

provided this limit exists and is approached uniformly. However, there are linear functionals which have no adjoints.

A large class of functionals are expressible in the form

$$L(f)=\iint_\Omega F(u, v, u_1, v_1)f(u_1, v_1)H(u_1, v_1)du_1dv_1 + \sum_{i,j=0}^{i+j=m} A_{ij}(u, v)\frac{\partial^{i+j}f(u, v)}{\partial u^i\partial v^j}. \tag{9}$$

The adjoint is

$$\overline{L}(f)=\iint_\Omega F(u_1, v_1, u, v)f(u_1, v_1)H(u_1, v_1)du_1dv_1 + \frac{1}{H}\sum_{i,j=0}^{i+j=m}(-1)^{i+j}\frac{\partial^{i+j}A_{ij}fH}{\partial u^i\partial v^j}. \tag{10}$$

This can be verified by substituting in equation (8), and applying Green's theorem repeatedly. If the function (9) is self-adjoint, the equation

$$F(u, v, u_1, v_1)=F(u_1, v_1, u, v)$$

will be satisfied, and also the equations obtained by equating the coefficients of $\partial^{k+l}/\partial u^k\partial v^l$ in the equation

$$\sum_{i,j=0}^{i+j=m} A_{ij}\frac{\partial^{i+j}f}{\partial u^i\partial v^j}=\frac{1}{H}\sum_{i,j=0}^{i+j=m}(-1)^{i+j}\frac{\partial^{i+j}A_{ij}Hf}{\partial u^i\partial v^j}.$$

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* Levy, *loc. cit.*, p. 116.

† Hadamard, "Lecons sur le calcul des variations," p. 303.

If $m=2$ these equations are seen to be

$$\left.\begin{aligned}
A_{20}&= \quad A_{20}, \quad A_{11}=A_{11}, \quad A_{02}=A_{02}, \\
A_{10}&=-A_{10}+\frac{1}{H}\left(2\frac{\partial A_{20}H}{\partial u}+\frac{\partial A_{11}H}{\partial v}\right), \\
A_{01}&=-A_{01}+\frac{1}{H}\left(\frac{\partial A_{11}H}{\partial u}+2\frac{\partial A_{02}H}{\partial v}\right), \\
A_{00}&= \quad A_{00}+\frac{1}{H}\left(\frac{\partial^2 A_{20}H}{\partial u^2}+\frac{\partial^2 A_{11}H}{\partial u\partial v}+\frac{\partial^2 A_{02}H}{\partial v^2}-\frac{\partial A_{10}H}{\partial u}-\frac{\partial A_{01}H}{\partial v}\right).
\end{aligned}\right\} \tag{11}$$

If the values of A_{10} and A_{01} are substituted in the last of these equations it becomes an identity. It follows that if A_{20}, A_{11}, A_{02} and A_{00} are taken arbitrarily, A_{10} and A_{01} can be determined so that the functional will be self-adjoint. It is evident from the equations similar to (11) for higher values of m, that m must be even, and that the functions A_{ij} with $i+j$ odd are determined by the others. It will be proved a little later that the functions A_{ij} with $i+j$ even can always be taken arbitrarily. The following theorems are easily verified.*

If $L(f)$ and $\bar{L}(f)$ are adjoint, then $h(u, v)L(k(uv)f)$ is adjoint to $k(u, v)\bar{L}(h(u, v)f)$, where $h(u, v)$ and $k(u, v)$ are arbitrary. It follows that if $L(f)$ is self-adjoint, $h(u, v)L(h(u, v)f)$ is also.

If $L(f)$ and $M(f)$ have the adjoints $\bar{L}(f)$ and $\bar{M}(f)$, respectively, then $L(M(f))$ is the adjoint of $\bar{M}(\bar{L}(f))$. Similarly, if $L(f)$ is self-adjoint, $\bar{M}(L(M(f)))$ is also.

From the last statement if $L(f)=\phi(u, v)f(u, v)$, where $\phi(u, v)$ is arbitrary, and $M(f)=\partial^{i+j}f(u, v)/\partial u^i\partial v^j$, then the functional

$$\bar{M}(L(M(f)))=\frac{(-1)^{i+j}}{H}\frac{\partial^{i+j}}{\partial u^i\partial v^j}\left(H\phi\frac{\partial^{i+j}f}{\partial u^i\partial v^j}\right) \tag{12}$$

must be self-adjoint. This may be expressed as

$$\sum_{k=i}^{2i}\sum_{l=j}^{2j}A_{kl}\frac{\partial^{k+l}f}{\partial u^k\partial v^l},$$

where $A_{2i2j}=(-1)^{i+j}\phi$. The function

$$\phi\frac{\partial^2 f}{\partial u\partial v}+\frac{1}{2H}\left(\frac{\partial\phi H}{\partial v}\cdot\frac{\partial f}{\partial u}+\frac{\partial\phi H}{\partial u}\cdot\frac{\partial f}{\partial v}\right)$$

* Compare with Levy, *loc. cit.*, p. 116.

is self-adjoint and can be used instead of $\phi(u, v)f$. It follows that

$$\bar{M}(L(M(f))) = \frac{(-1)^{i+j}}{H} \frac{\partial^{i+j}}{\partial u^i \partial v^j}\left(H\phi \frac{\partial^{i+j+2} f}{\partial u^{i+1} \partial v^{j+1}} + \frac{1}{2}\frac{\partial \phi H}{\partial v} \cdot \frac{\partial^{i+j+1} f}{\partial u^{i+1}\partial v^j} + \frac{1}{2}\frac{\partial \phi H}{\partial u} \cdot \frac{\partial^{i+j+1} f}{\partial u^i \partial v^{j+1}}\right) \quad (13)$$

is also self-adjoint. This is equal to

$$\sum_{k=i}^{2i+1} \sum_{l=j}^{2j+1} A_{kl} \frac{\partial^{k+l} f}{\partial u^k \partial v^l},$$

where $A_{2i+1\,2j+1} = (-1)^{i+j}\phi$, which is arbitrary. A self-adjoint functional of the type

$$\sum_{i,\,j=0}^{i+j=m} A_{ij} \frac{\partial^{i+j} f}{\partial u^i \partial v^j} \quad (14)$$

can then be formed by adding those of types (12) and (13), in which the functions A_{ij} with $i+j$ even are arbitrary, as has been stated previously. Since the functions A_{ij} with $i+j$ odd are determined uniquely by the condition that the functional be self-adjoint, every self-adjoint functional of type (14) is expressible as the sum of a finite number of functionals of types (12) and (13).

§ 3. *The Conditions that Given Functionals Be the Derivatives of Functions of Surfaces.*

In finding the variation of a functional depending on the surface S_λ defined by equations (4), the function $H(u, v)$ must be considered a function of λ also. Its derivative will now be calculated. By definition,

$$\frac{\partial H}{\partial \lambda} = \frac{1}{2H}\left(G\frac{\partial E}{\partial \lambda} + E\frac{\partial G}{\partial \lambda} - 2F\frac{\partial F}{\partial \lambda}\right), \quad (15)$$

and

$$\frac{\partial E}{\partial \lambda} = 2\Sigma x_u x_{u\lambda} = 2\Sigma(x_u X_u n_\lambda + x_u X n_{u\lambda}) = -2Dn_\lambda,$$

since

$$\Sigma x_u X = 0, \text{ and } \Sigma x_u X_u = -D.^*$$

If $\partial F/\partial\lambda$ and $\partial G/\partial\lambda$ are evaluated in the same way, equation (15) becomes

$$\frac{\partial H}{\partial \lambda} = -\frac{1}{H}(GD + ED'' - 2FD')n_\lambda = -K_m H n_\lambda,$$

where K_m is the mean curvature of the surface.†

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* The functions D, D' and D'' are called the fundamental coefficients of the second order. See Eisenhart, "Differential Geometry," p. 115.

† Eisenhart, *loc. cit.*, p. 123.

The condition that a given functional $\Psi_1(S;\ u,\ v)$ be the derivative of some function $\Psi(S)$ will now be derived by a method given by Volterra.* It will be assumed that Ψ_1 has a differential. That is to say that there is a linear functional $L(n_\lambda)$ such that $\partial\Psi_1/\partial\lambda = L(n_\lambda)$. A two parameter family of surfaces $S_{\lambda,\lambda'}$ will be determined by a function $n(u,\ v,\ \lambda,\ \lambda')$ having the properties assumed for $n(u,\ v,\ \lambda)$. Then a closed curve,

$$l: \qquad \lambda=\lambda(s), \quad \lambda'=\lambda'(s),$$

will be chosen bounding a small region ω in the λ, λ'-plane. It follows from Green's theorem that

$$\int_l ds \iint_\Omega \Psi_1(S_s;u,v) H\left(n_\lambda \frac{d\lambda}{ds} + n_{\lambda'}\frac{d\lambda'}{ds}\right) dudv$$
$$= \iint_\omega d\lambda d\lambda' \iint_\Omega \left\{\frac{\partial}{\partial\lambda}(\Psi_1 H n_{\lambda'}) - \frac{\partial}{\partial\lambda'}(\Psi_1 H n_\lambda)\right\} dudv.$$

This is equivalent to the equation

$$\int_l ds \iint_\Omega \Psi_1(S_s;u,v)\frac{dn}{ds} H dudv = \iint_\omega d\lambda d\lambda' \iint (L(n_\lambda) n_{\lambda'} - L(n_{\lambda'}) n_\lambda) H dudv. \quad (16)$$

If $\Psi_1(S;u,v)$ is the derivative of some function $\Psi(S)$, the left member of this equation is equal to

$$\int_l \frac{d\psi}{ds} ds,$$

which vanishes for every choice of n_λ and $n_{\lambda'}$. Consequently, the right member must vanish, and $L(n_\lambda)$ is self-adjoint. Conversely, if it is self-adjoint the right member will vanish. The value of Ψ can then be taken arbitrarily for one surface S_0, and for S_λ

$$\Psi(S_\lambda) = \Psi(S_0) + \int_0^\lambda d\lambda \iint_\Omega \Psi_1(S_\lambda;\ u,\ v) n_\lambda H dudv.$$

There are analogous conditions which must be satisfied if two given functionals $\Phi_n(n,f;u,v)$ and $\Phi_f(n,f;u,v)$ are to be equal to the partial derivatives of some function $\Phi(n,f)$. These functionals will be assumed to have differentials which satisfy the equations

$$\frac{\partial\Phi_n}{\partial\lambda} = \Phi_{nn}(n_\lambda) + \Phi_{nf}(f_\lambda), \quad \frac{\partial\Phi_f}{\partial\lambda} = \Phi_{fn}(n_\lambda) + \Phi_{ff}(f_\lambda). \quad (17)$$

* Volterra, "Fonctions de lignes," p. 45.

Equation (16) must then be replaced by the analogous equation

$$\int_l ds \iint_\Omega \left(\Phi_n \frac{dn}{ds} + \Phi_f \frac{df}{ds}\right) H du dv$$
$$= \iint_\omega d\lambda d\lambda' \iint_\Omega \{ (\Phi_{nn}(n_\lambda) + \Phi_{nf}(f_\lambda)) n_{\lambda'} + (\Phi_{fn}(n_\lambda) + \Phi_{ff}(f_\lambda)) f_{\lambda'}$$
$$- (\Phi_{nn}(n_{\lambda'}) + \Phi_{nf}(f_{\lambda'})) n_\lambda - (\Phi_{fn}(n_{\lambda'}) + \Phi_{ff}(f_{\lambda'})) f_\lambda + \Phi_f (f_{\lambda'\lambda} - f_{\lambda\lambda'})$$
$$- K_m \Phi_f (f_{\lambda'} n_\lambda - f_\lambda n_{\lambda'}) \} H du dv. \tag{18}$$

The derivatives $f_{\lambda'\lambda}$ and $f_{\lambda\lambda'}$ will not in general be equal. Their difference will be

$$f_{\lambda'\lambda} - f_{\lambda\lambda'} = \Sigma (f_u u_x + f_v v_x + f_n n_x)(x_{\lambda'\lambda} - x_{\lambda\lambda'}). \tag{19}$$

If the equations

$$u_x x_u + u_y y_u + u_z z_u = 1, \quad u_x x_v + u_y y_v + u_z z_v = 0, \quad u_x X + u_y Y + u_z Z = 0,$$

are solved for u_x, u_y and u_z, the equations

$$\Sigma u_x^2 = \frac{1}{H^2} \Sigma [(1 - X^2) x_v^2 - 2YZ y_v z_v] = \frac{1}{H^2} [G - (\Sigma X x_v)^2] = \frac{G}{H^2} \tag{20}$$

are easily derived. Similarly, it can be proved that

$$\Sigma u_x v_x = \frac{-F}{H^2}, \quad \Sigma v_x^2 = \frac{E}{H^2}, \quad n_x = X, \quad \Sigma u_x n_x = \Sigma v_x n_x = 0. \tag{21}$$

To evaluate $x_{\lambda'\lambda} - x_{\lambda\lambda'}$ equations (4) must be differentiated with respect to λ', and subtracted from the same equations with λ and λ' interchanged. This gives the equations

$$x_{\lambda'\lambda} - x_{\lambda\lambda'} = \frac{1}{H} \begin{vmatrix} Y y_v \\ Z z_v \end{vmatrix} \left(\frac{\partial n_\lambda}{\partial u} n_{\lambda'} - \frac{\partial n_{\lambda'}}{\partial u} n_\lambda \right) + \frac{1}{H} \begin{vmatrix} y_u Y \\ z_u Z \end{vmatrix} \left(\frac{\partial n_\lambda}{\partial v} n_{\lambda'} - \frac{\partial n_{\lambda'}}{\partial v} n_\lambda \right)$$
$$= -u_x \left(\frac{\partial n_\lambda}{\partial u} n_{\lambda'} - \frac{\partial n_{\lambda'}}{\partial u} n_\lambda \right) - v_x \left(\frac{\partial n_\lambda}{\partial v} n_{\lambda'} - \frac{\partial n_{\lambda'}}{\partial v} n_\lambda \right).$$

If these values are substituted in equation (19) it can be reduced to the form

$$f_{\lambda'\lambda} - f_{\lambda\lambda'} = \frac{1}{H^2} \left\{ (-f_u G + f_v F) \left(\frac{\partial n_\lambda}{\partial u} n_{\lambda'} - \frac{\partial n_{\lambda'}}{\partial u} n_\lambda \right) + (f_u F - f_v E) \left(\frac{\partial n_\lambda}{\partial v} n_{\lambda'} - \frac{\partial n_{\lambda'}}{\partial v} n_\lambda \right) \right\}.$$

If this expression is substituted in equation (18) it becomes

$$\int_l ds \iint_\Omega \left(\Phi_n \frac{dn}{ds} + \Phi_f \frac{df}{ds}\right) H du dv = \iint_\omega d\lambda d\lambda' \iint_\Omega \left\{ \left[\Phi_{nn}(n_\lambda) \right.\right.$$
$$\left. + \frac{\Phi_f}{H^2} \left((-f_u G + f_v F) \frac{\partial n_\lambda}{\partial u} + (f_u F - f_v E) \frac{\partial n_\lambda}{\partial v} \right) \right] n_{\lambda'}$$
$$- \left[\Phi_{nn}(n_{\lambda'}) + \frac{\Phi_f}{H^2} \left((-f_u G + f_v F) \frac{\partial n_{\lambda'}}{\partial u} + (f_u F - f_v E) \frac{\partial n_{\lambda'}}{\partial v} \right) \right] n_\lambda$$
$$+ [\Phi_{nf}(f_\lambda) + K_m \Phi_f f_\lambda] n_{\lambda'} - \Phi_{fn}(n_{\lambda'}) f_\lambda + \Phi_{fn}(n_\lambda) f_{\lambda'}$$
$$\left. - [\Phi_{nf}(f_{\lambda'}) + K_m \Phi_f f_{\lambda'}] n_\lambda + \Phi_{ff}(f_\lambda) f_{\lambda'} - \Phi_{ff}(f_{\lambda'}) f_\lambda \right\} H du dv.$$

If there is a function $\Phi(n, f)$ which has the partial derivatives $\Phi'_n = \Phi_n$ and $\Phi'_f = \Phi_f$ the left member of the last equation, and consequently the right member, must vanish for every choice of the functions n_λ, $n_{\lambda'}$, f_λ and $f_{\lambda'}$. The necessary and sufficient conditions for this are that the functionals

$$\Phi_{nn}(g) + \frac{\Phi_f}{H^2}\left((-f_u G + f_v F)\frac{\partial g}{\partial u} + (f_u F - f_v E)\frac{\partial g}{\partial v}\right)$$

and $\Phi_{ff}(g)$ be self-adjoint, and that $\Phi_{nf}(g) + K_m \Phi_f g$ be the adjoint of $\Phi_{fn}(g)$. The similar conditions for functions of lines are given by Levy.*

§ 4. *Equations Involving Partial Functional Derivatives.*

The condition of integrability of an equation such as

$$\Phi'_n(n, f; u, v) = W(n, f, \Phi'_f; \Phi, u, v) \tag{22}$$

will now be found.† If there is an integral $\Phi(n, f)$ which is equal to an arbitrarily given function $\Psi(f)$ when $n=0$, the equation is said to be completely integrable. It will be assumed that the derivative $\Psi'(f; u, v)$ is continuous and approached uniformly, and that Ψ' and W have differentials, implying the equations

$$\frac{d\Psi'}{d\lambda} = \Psi_{ff}(f_\lambda), \quad \frac{dW}{d\lambda} = Q\left(\frac{d\Phi'_f}{d\lambda}\right) + M(n_\lambda) + L(f_\lambda), \tag{23}$$

for every choice of the functions $n(u, v, \lambda)$ and $f(u, v, \lambda)$ such that f_λ, n_λ and $\partial\Phi'_f/\partial\lambda$ are of class $C^{(r)}$ in u and v.

If equation (22) is differentiated with respect to λ, and values substituted from equations (17) and (23), it becomes

$$\Phi_{nn}(n_\lambda) + \Phi_{nf}(f_\lambda) = Q(\Phi_{fn}(n_\lambda)) + Q(\Phi_{ff}(f_\lambda)) + M(n_\lambda) + L(f_\lambda).$$

It follows that for any function $g(u, v)$,

$$\Phi_{nn}(g) = Q(\Phi_{fn}(g)) + M(g),$$

and

$$\Phi_{nf}(g) = Q(\Phi_{ff}(g)) + L(g).$$

On the surface defined by $n=0$, the functional Φ_{ff} is equal to Ψ_{ff} which must be self-adjoint. The functional $\Phi_{fn}(g)$ is the adjoint of $\Phi_{nf}(g) + K_m\Phi_f g$, and, consequently,

$$\Phi_{fn}(g) = \Phi_{ff}(\bar{Q}(g)) + \bar{L}(g) + K_m \Phi'_f g. \tag{24}$$

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* Levy, *loc. cit.*, p. 120.

† Compare with Levy, *loc. cit.*, pp. 122–123.

The other functional which was proved to be self-adjoint in the last section is now equal to

$$Q(\Phi_{ff}(\bar{Q}(g)))+Q(\bar{L}(g))+Q(K_m\Phi'_f g)+M(g)$$
$$+\frac{\Phi'_f}{H^2}\left((-f_uG+f_vF)\frac{\partial g}{\partial u}+(f_uF-f_vE)\frac{\partial g}{\partial v}\right).$$

The first term is self-adjoint as was proved in § 2. A necessary condition that equation (22) be completely integrable is, therefore, that the functional

$$Q(\bar{L}(g))+Q(K_m\Phi'_f g)+M(g)+\frac{\Phi'_f}{H^2}\left((-f_uG+f_vF)\frac{\partial g}{\partial u}+(f_uF-f_vE)\frac{\partial g}{\partial v}\right) \quad (25)$$

be self-adjoint.

To illustrate the theory just developed, the equation will be found which must be satisfied by the function

$$\Phi=\int\int\int_R(f_x^2+f_y^2+f_z^2)\,dxdydz, \quad (26)$$

where $f(x, y, z)$ is a solution of the equation

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}=0. \quad (27)$$

The value of f inside the region R is then determined by the values it takes on the surface S which bounds R. If S is considered as fixed, while the boundary values of $f(x, y, z)$ are varied, the equation

$$\frac{d\Phi}{d\lambda}=2\int\int\int_R \Sigma f_x f_{x\lambda}\,dxdydz$$

will be satisfied. This can be reduced by means of Green's formula and equation (27) to the form

$$\frac{d\Phi}{d\lambda}=2\int\int_S f_n f_\lambda H\,dudv.$$

It follows that

$$\Phi'_f(n, f; u, v)=2f_n(x(u, v), y(u, v), z(u, v)). \quad (28)$$

If the value of $f(x, y, z)$ is fixed at each point of R, while S is varied, the boundary values of $f(x, y, z)$ will vary according to the law $f_\lambda=f_n n_\lambda$. In this case it follows from equations (7) and (26) that

$$\int\int(\Phi'_n+f_n\Phi'_f)n_\lambda H\,dudv=\int\int(f_x^2+f_x^2+f_z^2)n_\lambda H\,dudv,$$

and, consequently,

$$\Phi'_n+f_n\Phi'_f=\Sigma f_x^2.$$

If f_x is replaced by $f_u u_x+f_v v_x+f_n n_x$ and equations (20) and (21) applied, the right member of this equation becomes

$$\frac{1}{H^2}(f_u^2G-2f_uf_vF+f_v^2E)+f_n^2.$$

Substituting the value of f_n from equation (28),

$$\Phi'_n = -\tfrac{1}{4}\Phi_f'^2 + \frac{1}{H^2}\,(f_u^2 G - 2 f_u f_v F + f_v^2 E).$$

This is the desired equation.

It can easily be proved that the condition of integrability is satisfied. The functionals Q, M and L are, evidently,

$$Q(g) = -\tfrac{1}{2}\Phi'_f g, \quad M(g) = 0,$$

$$L(g) = \frac{2}{H^2}\left[(f_u G - f_v F)\frac{\partial g}{\partial u} + (-f_u F + f_v E)\frac{\partial g}{\partial v}\right].$$

If these values are substituted in the expression (25) the terms involving $\frac{\partial g}{\partial u}$ and $\frac{\partial g}{\partial v}$ cancel and the other terms are self-adjoint.

§ 5. *Characteristics.*

There is a set of functionals which have the same relation to a solution of equation (22) that the characteristics have to a solution of a partial differential equation. The characteristics of equations involving partial derivatives of functions of lines have been discussed by Levy.* There are characteristics of various orders, but the most interesting ones are the "caractéristiques de première espèce," and these are the only ones which will be considered here. After they are defined it will be proved that if an integral contains an element of a characteristic, it contains the whole characteristic. An element and an integral will be defined first.

An element is any set of functions $n(u, v)$, $f(u, v)$, $\Phi(n, f)$, $\Phi'_n(n, f; u, v)$ and $\Phi'_f(n, f; u, v)$ which satisfy equation (22).

An integral is a function $\Phi(n, f)$ whose partial derivatives satisfy equation (22) for every pair of admissible functions $n(u, v)$ and $f(u, v)$.

A characteristic is a set of functions $g(u, v)$, $\Psi(n)$, $\Psi'_n(n; u, v)$ and $\Psi'_f(n; u, v)$ which satisfy the equations

$$\frac{dg}{d\lambda} = -\bar{Q}(n_\lambda),$$

$$\frac{d\Psi}{d\lambda} = \iint_\Omega (\Psi'_n - Q(\Psi'_f))\, n_\lambda H\, du\, dv, \tag{29}$$

$$\frac{d\Psi'_n}{d\lambda} = Q(\bar{L}(n_\lambda)) + Q(K_m \Psi'_f n_\lambda) - L(\bar{Q}(n_\lambda)) + M(n_\lambda), \tag{30}$$

$$\frac{d\Psi'_f}{d\lambda} = \bar{L}(n_\lambda) + K_m \Psi'_f n_\lambda, \tag{31}$$

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* Levy, *loc. cit.*, Ch. II.

where the function Ψ is used instead of Φ in deriving Q, L and M. If $\Phi(n, f)$ is an integral the equations

$$\frac{d\Phi}{d\lambda}=\int\int_{\Omega}(\Phi'_n n_\lambda+\Phi'_f f_\lambda)H du dv, \tag{32}$$

$$\frac{d\Phi'_n}{d\lambda}=Q\left(\frac{d\Phi'_f}{d\lambda}\right)+L(f_\lambda)+M(n_\lambda), \tag{33}$$

and

$$\frac{d\Phi'_f}{d\lambda}=\Phi_{ff}(f_\lambda)+\Phi_{ff}(\bar{Q}(n_\lambda))+\bar{L}(n_\lambda)+K_m\Phi'_f n_\lambda, \tag{34}$$

will be satisfied, on account of equations (7), (17), (22), (23) and (24). If the value given for g_λ is substituted for f_λ in equations (32), (33) and (34), they become equivalent to (29), (30) and (31). It follows that if the integral $\Phi(n, g)$ and its derivatives are equal to $\Psi(n)$, $\Psi'_n(n; u, v)$ and $\Psi'_f(n; u, v)$, respectively, for $\lambda=\lambda_0$, they are equal for all values of λ for which the equations of the characteristics have unique solutions.

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